

[P5] Lower bounds for $g(r)$ at every r : $3r - 3$, and $3r - 2$ from $r = 16$

Mohamed A. Osman — ORCID 0009-0004-5912-999X

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Abstract

Let $g(r)$ be the least number of hyperedges in an intersecting r -uniform hypergraph with transversal number r . Sivashankar [2] proved $g(r) \geq 3r - 4$ and asked whether his residual bound $4\tau \leq q + r + 4$ can be improved to $+3$. We give a proposed proof that it can (Theorem 1), so $g(r) \geq 3r - 3$ for every r ; this is attained at $r = 3$ and $r = 4$. We then exclude equality for every $r \geq 16$ (Theorem 2), so $g(r) \geq 3r - 2$ there. Every finite step is checked exhaustively by a named script. The asymptotic bound $g(r) \geq (3.1108 - o(1))r$ is in the companion paper [P6].

1. Setting

1.1 The problem

A hypergraph H is **intersecting** if any two hyperedges meet, and $\tau(H)$ is the least number of vertices meeting every hyperedge. Then

$$g(r) = \min\{|E(H)| : H \text{ intersecting, } r\text{-uniform, } \tau(H) = r\}.$$

The problem goes back to Erdős and Lovász [1], and Kahn [3] proved $g(r) = O(r)$. The known values are $g(3) = 6$ and $g(4) = 9$ [5], $g(5) = 13$ [4] and $g(6) = 17$ ([P1]–[P4] of this set). Against $3r$ these are $3r - 3$, $3r - 3$, $3r - 2$ and $3r - 1$. For $r = 7$, an explicit family of 21 hyperedges with $\tau = 7$, found by a computer search among families invariant under a cyclic group of order 7, gives $g(7) \leq 21$ (data/witness_r7_21.json); with Theorem 1, $18 \leq g(7) \leq 21$. Whether that upper bound is already known has not been checked.

The reduction of [2]. Repeatedly take a vertex of degree at least four and delete the hyperedges through it. If this takes k steps and leaves a family J of q hyperedges and maximum degree at most three, then

$$|E(H)| \geq q + 4k, \quad r \leq k + \tau(J).$$

So a bound $4\tau(J) \leq q + r + C_0$ gives $|E(H)| \geq 3r - C_0$. Theorem 1 is this bound with $C_0 = 3$.

1.2 The remainder J

We use the dual picture: each hyperedge of J is a **point**, and each vertex is a **block**, the set of points containing it. Since J is intersecting, r -uniform and of maximum degree three:

- every block has at most three points, and every pair of points lies in some block;
- every point lies in exactly r blocks (two vertices with the same block count twice).

A point A meets the other $q - 1$ points through its r blocks, each of which reaches at most two of them. So

$$q - 1 \leq \sum_{v \in A} (\deg_J v - 1) \leq 2r. \quad (1.1)$$

Triples, W and U . Fix a maximum family of pairwise disjoint 3-point blocks $M_1, \dots, M_t$. Let W be their union ($3t$ points) and U the other u points, so $q = 3t + u$. No 3-point block lies inside U , by maximality.

The natural cover. One vertex for each M_i , one common vertex for each pair of a pairing of U , and one more vertex if u is odd, cover J . So

$$\tau(J) \leq t + \lceil u/2 \rceil. \quad (1.2)$$

Witnesses. For each pair $x, y \in U$ fix one common vertex w_{xy} , the *witness* of the pair. Its block is $\{x, y\}$ or $\{x, y, z\}$ with $z \in W$. Different pairs have different witnesses, since a repeated witness would give a block of four points or a 3-point block inside U .

Local graphs. For each i , the *local graph* F_i is the graph on U with an edge xy of colour $z \in M_i$ whenever the witness block is $\{x, y, z\}$. Two disjoint edges of F_i always have the same colour, since otherwise their blocks would replace M_i by two disjoint triples.

Notation used throughout.

- For one colour: e = its number of edges, s = its support (vertices it touches).
- S_i = sum of the three colour supports of F_i , and $S = \sum_i S_i$.
- S_x = number of points of W reached by the witnesses at $x \in U$; then $S = \sum_x S_x$.
- $d = 2r - q + 1$, the slack in (1.1).

2. Theorem 1: the sharp residual bound

Theorem 1 (proposed). *Every intersecting r -uniform family J with q hyperedges and maximum degree at most three satisfies $4\tau(J) \leq q + r + 3$.*

Corollary 1. $g(r) \geq 3r - 3$ for every $r \geq 1$.

Proof. The reduction of §1.1 with Theorem 1. $\square$

2.1 Capacity and local support

Capacity at U . The witnesses use exactly $u(u-1)$ of the ur incidences of points of U . Each of the other $ur - u(u-1)$ incidences reaches at most two points of W , and all $3tu$ pairs between U and W must be met. Hence

$$3tu - S \leq 2(ur - u(u-1)). \quad (2.1)$$

Capacity of a colour. Fix a colour $z \in M_i$ with e edges and support s . In the point z , the e witnesses cover s points of U , the vertex of M_i covers its two other points, and the remaining $q - 3 - s$ points need the other $r - e - 1$ vertices, at most two each. So

$$2e - s \leq 2r - q + 1 = d. \quad (2.2)$$

Lemma 1 ([2] for the first bound). *Let F be a simple graph on u vertices with at most three colours, in which disjoint edges have the same colour. Then $S_F \leq \max(12, u + 4)$. If every colour has $2e - s \leq d$, then $S_F \leq \max(12, u + 2, d + 8)$.*

Proof. Suppose first that some colour has two disjoint edges, with endpoint set Y of four vertices. Every edge of another colour meets both, so it lies in Y .

- If no other colour is present, $S_F \leq u$.
- If the other-colour edges have no common vertex, every first-colour edge meets all of them, so it lies in Y too, and $S_F \leq 12$.
- Otherwise there are at most two other-colour edges, sharing a vertex, contributing at most four: $S_F \leq u + 4$. Fix one of them, g . Every first-colour edge meets g , so $e \geq s - 2$, and $2e - s \leq d + 4$ gives $s \leq d + 4$, hence $S_F \leq d + 8$.

Suppose next that no colour has two disjoint edges. Then each colour is a star or part of a triangle. If all supports are at most four, $S_F \leq 12$. Otherwise one colour is a star with at least four leaves; every other-colour edge contains its centre, and simplicity gives $S_F \leq u + 2$. $\square$

2.2 A preliminary bound

Lemma 2 ([2]). $r \geq u + t - 2$.

Proof. Put $\alpha = r - u - t + 2$. If $u = 0$, (1.1) gives $\alpha \geq (r + 5)/3 > 0$. Otherwise (1.1), (2.1) and Lemma 1, with $K = \max(12, u + 4)$, give

$$t \leq u - 3 + 2\alpha, \quad \alpha \geq 1 - \frac{(K - u)t}{2u}.$$

A negative integer α would give $t \leq u - 5$ and then $\alpha > -1$. For $K = u + 4$ the bound is $-1 + 10/u$; for $K = 12$ with $u \leq 8$ it needs $u^2 - 13u + 60 > 0$, whose discriminant is negative. $\square$

2.3 The odd equality case

By (1.2) and Lemma 2:

- if u is even, $4\tau \leq 4t + 2u = q + (t + u) \leq q + r + 2$;
- if u is odd, $4\tau \leq q + (t + u) + 2$.

So Theorem 1 follows once we show: *if u is odd then $r \geq u + t - 1$* . Suppose u is odd and $r = u + t - 2$. Then $d = u - t - 3 \geq 0$, and (2.1) becomes

$$S \geq u(t + 2). \tag{2.3}$$

Large u . If $u \geq 11$, (2.3) and Lemma 1 give $t \geq u/2$. Then $d + 8 \leq u + 2$ and $12 \leq u + 2$, so every $S_i \leq u + 2$ and $u(t + 2) \leq t(u + 2)$, that is $u \leq t$, against $t \leq u - 3$.

Small u . For odd $u \leq 9$, the conditions $0 \leq t \leq u - 3$ and $u(t + 2) \leq t \max(12, u + 4)$ leave exactly

$$(u, t) \in \{(5, 2), (7, 3), (7, 4), (9, 5), (9, 6)\},$$

with $d = 0, 1, 0, 1, 0$ respectively.

Lemma 3 (small excess). *If every colour has $2e - s \leq 1$ and F has more than one colour, then $S_F \leq 12$, $S_F \neq 11$, and $S_F = 12$ only when F is a properly three-coloured K_4 . If also $u \leq 9$, the same holds when F has one colour.*

Proof. A connected component on $v \geq 2$ vertices has $2e - v \geq v - 2$, so each colour is a matching, or one path on three vertices plus a matching.

- A colour with a matching of size three leaves the other colours empty, so F has one colour and $S_F \leq u \leq 9$.

- A colour of support five is P_3 plus an edge; the other colours are then confined to two edges, and $S_F \leq 9$.
- Otherwise every support is at most four. Support four forces two disjoint edges, and two cross-intersecting two-edge matchings are two perfect matchings of one K_4 ; a third colour must be the third. $\square$

The cases $d = 1$.

- $(9, 5)$: (2.3) asks $S \geq 63$, and Lemma 3 allows 60.
- $(7, 3)$: (2.3) asks 35, so all three S_i are 12, that is, three edge-disjoint copies of K_4 on seven vertices. Two such copies meet in one vertex and leave $K_{3,3}$ plus an isolated vertex, which holds no third K_4 .

The cases $d = 0$. Now every colour is a matching and $q = 2r + 1$, so (1.1) is an equality for every point: every vertex has degree three and any two points meet once. So every witness block is a triple, and the F_i partition the edges of K_u .

- $(5, 2)$: $qr = 55$ is not divisible by three.
- For $u \leq 9$ a local graph has at most six edges: one colour is a matching of at most four edges, and with several colours each is a matching of at most two, since a third disjoint edge would meet no other-colour edge. Six edges occur only as the three perfect matchings of a K_4 , and five only as K_4 minus an edge.
- $(9, 6)$: the 36 edges of K_9 would split into six copies of K_4 , which needs the degree 8 to be divisible by three.
- $(7, 4)$: the 21 edges of K_7 split into four parts of size at most six only as $6 + 6 + 6 + 3$, $6 + 6 + 5 + 4$ or $6 + 5 + 5 + 5$. Two edge-disjoint K_4 leave $K_{3,3}$ plus a vertex, which excludes the first two. In the third, fix the K_4 on four vertices. Each copy of K_4 minus an edge contains a triangle, so it takes one of the three edges among the other three vertices; it is then two of the four joined to two of the three, plus that edge. The three pairs taken among the three are distinct and meet, so edge-disjointness needs three disjoint pairs among four vertices, which is impossible.

This proves Theorem 1. $\square$

2.4 Sharpness, and degree four

For odd n take points $1, \dots, n$ and one vertex for each pair $([2])$. Then $q = n$, $r = n - 1$, a cover is an edge cover of K_n , and $\tau = (n + 1)/2$, so $4\tau = q + r + 3$.

Corollary 2. *If J has maximum degree at most four, then $4\tau(J) \leq q + r + 3$.*

Proof. Take a maximal family of disjoint 4-point blocks, a of them. Their vertices cover $4a$ points and the rest has maximum degree three, so $\tau \leq a + (q - 4a + r + 3)/4$. $\square$

Peeling at degree five then gives $|E(H)| \geq (15r - q - 15)/4$ for a remainder of size q . This improves the additive term in the corresponding step of [2], not its coefficient.

3. The equality case

In Sections 3 and 4 suppose $|E(H)| = 3r - 3$ and $\tau(H) = r$ with $r \geq 5$. We write *deleted hyperedges* for the hyperedges removed in the reduction.

3.1 The parameters

The maximum degree of H is four:

- a vertex of degree at least five, peeled first, would give $|E(H)| \geq |E(J)| + 4(r - \tau(J)) + 1 \geq 3r - 2$ by Theorem 1;
- if every degree were at most three, $3r - 3 \leq 2r + 1$ would force $r \leq 4$. (At $r = 3$ and $r = 4$ the bound $3r - 3$ is attained.)

Take a maximum family of disjoint 4-point blocks, a of them. Their $4a$ hyperedges form *groups* of four, each with its *group vertex*. Delete them and let J be the rest: maximum degree three and $q = 3r - 3 - 4a$. The a group vertices and a cover of J cover H , so $\tau(J) \geq r - a$, while Theorem 1 gives $4\tau(J) \leq q + r + 3 = 4(r - a)$. So Theorem 1 is an equality for J , and §2.3 forces u odd, $\tau(J) = r - a$, $r = u + t - 1$ and $q = 3t + u$. Consequently

$$u = 2a + 3, \quad t = r - 2a - 2, \quad d = u - t - 1 \geq 0, \quad r = 2u - 2 - d, \quad (3.1)$$

and (2.1) becomes $S \geq ut$. Write $H_0 = (u - 1)/2$.

When $d = 0$ every vertex of a point of J has degree three in J ; as d grows, vertices of degree two and one are allowed.

3.2 Local bounds

Lemma 4 (pointwise capacity). *For every $x \in U$, $S_x \geq t$. In one F_i , if x carries E edges in k colours, then $E - k \leq d$.*

Proof. The point x has exactly $u - 1$ witnesses, so exactly $r - (u - 1) = t$ other vertices. They meet at most $2t$ points of W , and all $3t$ must be met, so $3t - S_x \leq 2t$. The $u - 1$ witnesses at x reach S_x distinct points of W , so the number of witnesses *wasted* at x is $u - 1 - S_x \leq u - 1 - t = d$. Edges of one colour at x reach one point, so E edges in k colours waste $E - k$. $\square$

Lemma 5. *Every F_i with more than one colour has $S_i \leq \max(12, d + 8)$ and $S_i - e(F_i) \leq 6$. A one-coloured F_i has $S_i - e(F_i) \leq H_0$.*

Proof. The first case of Lemma 1 gives $\max(12, d + 8)$. In the star case, F_i is a star at one vertex with E edges in $k \leq 3$ colours, and Lemma 4 gives $S_i = E + k \leq d + 2k \leq d + 6$.

For $S_i - e(F_i)$: if a colour has two disjoint edges on Y , either every support lies in Y and each colour has $s - e \leq 2$, or there are at most two other-colour edges, contributing at most two, while the first colour has $e \geq s - 2$. Otherwise every colour is a star or a triangle, with $s - e \leq 1$. A one-coloured graph has $s - e$ at most its number of components, at most H_0 . $\square$

3.3 Two augmentation bounds

Call a one-coloured F_i *large* if its matching number is at least four, and let m be the number of large ones.

Lemma 6. $2m \leq t$.

Proof. A large F_i uses one colour z_i , so the other two points of M_i are met by no witness. Every $x \in U$ must therefore meet these $2m$ points through its t non-witness vertices, and no such vertex meets two of them:

- within one M_i , its block and an edge of F_i avoiding x would replace M_i by two triples;

- across M_i and M_j , its block, an edge of F_i avoiding x , and an edge of F_j avoiding x and that edge (matching number four provides both) would give three triples in place of two.

Either way the family of triples was not maximum. $\square$

Lemma 7. $u \leq \max(12, d + 8)$.

Proof. Suppose not. By Lemma 5 every multicoloured F_i has $S_i < u$, and a one-coloured one has $S_i \leq u$. Since $S \geq ut$, every F_i is one-coloured and spans U . Then $2e - u \leq d$ gives $e \leq (u + d)/2$, so its matching number is at least $u - e \geq (u - d)/2 > 4$. So $m = t$, and Lemma 6 forces $t = 0$, whereas $t = u - d - 1 \geq 8$. $\square$

3.4 The swap lemma

The groups come from a *maximum* family of 4-point blocks. That alone constrains every deleted hyperedge.

Lemma 8 (swap). *Let $A \neq A'$ be deleted hyperedges of the same group. If $w \in A$ and $w' \in A'$ are vertices of degree three in J , their sets of points of J meet.*

Proof. The vertex w lies in A and in three points of J , so its block is $\{A\} \cup T$ with T three points of J ; likewise $\{A'\} \cup T'$. Neither block meets another group. If $T \cap T' = \emptyset$, the two blocks are disjoint, and putting them in place of the group's own block gives $a + 1$ disjoint 4-point blocks, against maximality. $\square$

Call such a T a *triple* of A . A deleted hyperedge meets J through its triples and through vertices of degree two or one in J .

Proposition 1. *In the equality setting $d \neq 0$, and $d \neq 1$ when $r \geq 11$.*

Proof. Fix a group of four deleted hyperedges.

$d = 0$. By §2.3, $q = 2r + 1$ and every vertex of a point of J has degree three in J . So a deleted hyperedge A meets the $2r + 1$ points of J through at most $r - 1$ triples. Fix one, T . If the other at most $r - 2$ triples all met T , they would reach at most $3 + 2(r - 2) = 2r - 1$ points; so A has a triple T' disjoint from T . By Lemma 8 every triple of a group-mate A' meets both T and T' , so it reaches at most one of the $2r - 5$ points outside them. With at most $r - 1$ triples, A' misses one, since $r - 1 < 2r - 5$ for $r \geq 5$.

$d = 1$. Now $q = 2r$, and (1.1) allows each point of J at most one vertex of degree two and none of degree one. So J has at most $q/2 = r$ vertices of degree two, each in at most two deleted hyperedges: at most $2r$ incidences with deleted hyperedges.

- A deleted hyperedge whose triples pairwise meet reaches at most $1 + 2n_3$ points through its n_3 triples and two through each other vertex, at most $1 + 2(r - 1) < 2r$ in all. So every deleted hyperedge has two disjoint triples.
- For a group-mate A' with n_2 vertices of degree two, the $2r - 6$ points outside those two triples are reached at most once per triple and twice per vertex of degree two, so $2r - 6 \leq (r - 1) + n_2$ and $n_2 \geq r - 5$.

Every deleted hyperedge is such a group-mate, so the $4a \geq 4$ deleted hyperedges need at least $4(r - 5)$ incidences, more than $2r$ when $r \geq 11$. $\square$

4. Theorem 2: excluding equality from $r = 16$

Theorem 2 (proposed). $g(r) \geq 3r - 2$ for every $r \geq 16$.

4.1 Traces of the deleted hyperedges

For a deleted hyperedge A , its *trace* L_A is the graph on U with an edge xy whenever $w_{xy} \in A$. Write e_A , s_A , ν_A for its number of edges, its support and its matching number. Put

$$T = \sum_i e(F_i), \quad D = S - T, \quad M = \sum_A (u - s_A).$$

- A witness of degree two in J lies in at most two deleted hyperedges, and one of degree three in at most one, since degrees in H are at most four. So $\sum_A e_A \leq u(u-1) - T$.
- For $x \in U$, its t non-witness vertices have room for at most $3t$ further hyperedges. They must meet the $3t - S_x$ points of W missed by the witnesses, and every deleted A whose trace misses x . So at most S_x deleted hyperedges miss x , and $M \leq S$.

With $\nu_A \geq s_A - e_A$ and $4a = 2u - 6$ this gives

$$\sum_A \nu_A \geq u^2 - 5u - D. \quad (4.1)$$

Lemma 9. *Within one group there are no three disjoint edges from three different traces.*

Proof. Extend them to a matching of H_0 edges of the complete graph on U , leaving one point x . The witnesses of these edges cover U except x and meet three hyperedges of the group; one vertex of x that lies in the fourth covers both. With the t vertices of the triples and the $a - 1$ other group vertices this is a cover of H of size $H_0 + 1 + t + a - 1 = r - 1$. $\square$

Lemma 10. *Three traces each with a matching of size four, and a fourth nonempty trace, contain three disjoint edges from three different traces.*

Lemma 11. *The same holds for four traces with matching numbers at least 3, 3, 4, 4.*

Proof of both. Fix size-four matchings P, Q in two of the traces and suppose no such triple exists. Call an edge e of another trace *blocked* if no two disjoint edges, one from P and one from Q , avoid it; every edge of the other traces is blocked.

Step 1: each blocked edge sits on a four-cycle. The edges of P and of Q avoiding a blocked edge must cross-intersect. Each side keeps at least two, and one edge cannot meet three disjoint edges, so exactly two survive on each side. They form a four-cycle, which is a component of $P \cup Q$ (counting an edge twice if it is in both). Both endpoints of e lie in both supports, since e removes exactly two edges from each matching.

Step 2: two disjoint four-cycles. The other traces supply three disjoint blocked edges (a size-three matching for Lemma 11, size four for Lemma 10). Their six endpoints lie in both supports, and outside one such four-cycle there are at most four such vertices, so another blocked edge touches it. The four-cycle of that edge avoids it, so it is a second, disjoint four-cycle. Hence $P \cup Q$ is two disjoint four-cycles on the same eight vertices, and every blocked edge is a diagonal of one of them.

Step 3: a rainbow triple. For Lemma 11, the two size-three matchings each consist of three of the four diagonals. The first has both diagonals of one cycle and the second at least one of them; take that one for the second, the other for the first, and an edge of P in the other cycle. For Lemma

10, the three size-four matchings are the three perfect matchings of each of the two copies of K_4 . Take an edge f of the fourth trace. If f removes two vertices of one copy, the edge on its other two vertices and an edge of a different matching in the other copy avoid f . If f removes at most one vertex of each copy, each copy keeps a triangle or all of K_4 , so two edges of different matchings avoid f . $\square$

Order the matching numbers of one group as $\nu_1 \leq \nu_2 \leq \nu_3 \leq \nu_4$. Greedy choice gives a rainbow triple when $\nu_2 \geq 1$, $\nu_3 \geq 3$ and $\nu_4 \geq 5$. With Lemmas 9–11, every group has $\nu_2 = 0$, or $\nu_3 \leq 2$, or all $\nu_i \leq 4$ with neither $\nu_1 \geq 1$, $\nu_2 = \nu_3 = \nu_4 = 4$ nor $\nu_1, \nu_2 \geq 3$, $\nu_3 = \nu_4 = 4$. So the sum over one group is at most $\max(2H_0, H_0 + 6, 13)$, and with (4.1)

$$D \geq \frac{u^2 - 6u - 3}{2} \quad (u \geq 15), \quad D \geq 39 \quad (u = 13). \quad (4.2)$$

4.2 Large remainders: u is at most 11

$u \geq 15$. By Lemma 7, $d \geq u - 8$ and $t = u - d - 1 \leq 7$. By Lemma 5, $D \leq 7H_0 = 7(u - 1)/2$. With (4.2) this gives $u^2 - 13u + 4 \leq 0$, false for $u \geq 13$.

$u = 13$. Lemma 7 gives $d \geq 5$.

- If $d \geq 6$ then $t \leq 6$ and $D \leq 6t \leq 36$, against (4.2).
- If $d = 5$ then $t = 7$ and every $S_i \leq 13$, so $S \geq 91$ forces every $S_i = 13$. A one-coloured F_i then has $e \leq 9$ and matching number at least four, so it is large and $S_i - e \leq 6$. A multicoloured F_i cannot be the star case of Lemma 5 (which gives at most $d + 6 = 11$), so $S_i = 13 = d + 8$ is the extremal case of Lemma 1: two other-colour edges pa , pb , and a first colour made of ab and six edges at p , so $S_i - e = 13 - 9 = 4$. Lemma 6 gives $m \leq 3$, so $D \leq 6 \cdot 3 + 4 \cdot 4 = 34$, against (4.2).

So $u \leq 11$.

4.3 $u = 11$: the anchor count

Open pairs. A pair (x, c) is *open* in a local graph F_j if no edge of colour c meets x and every edge of the other two colours does.

Anchor value. For a local graph F_i and $x \in U$, let P_x be the set of colours of F_i with no edge at x ; their points of M_i are not reached by the witnesses at x .

- A pair of colours in P_x is *forbidden* if F_i has an edge of the third colour avoiding x .
- A colour $c \in P_x$ is *good* if, for every pair f of U that avoids x and is not an edge of F_i , some edge of F_i of a colour other than c avoids both x and f .

For a set $R \subseteq P_x$ whose pairs are all forbidden, let g_x be $|R|$, minus the number of colours of P_x outside R , minus the number of colours of R that are not good; take the best R . The *anchor value* of F_i is $G = \sum_x \max(g_x, 0)$.

Lemma 12 (demand). *If F_i has anchor value G , the other local graphs have at least $tu + G - S$ open pairs in all.*

Proof. Fix x and R . The t non-witness vertices of x must reach the $3t - S_x$ points of W that its witnesses miss, among them the points of R . No vertex reaches two points of R : its block and an edge of the third colour avoiding x would replace M_i by two triples. So $|R|$ distinct vertices reach

the points of R , each reaching at most one further point, and the other $t - |R|$ reach at most two each. The $|R|$ vertices therefore reach at least $t + |R| - S_x$ further points missed by the witnesses, of which at most one per colour of P_x outside R lies in M_i .

Let $\{x, c, q'\}$ be the block of such a vertex, with $c \in R$ good and $q' \in M_j$, $j \neq i$. No edge of colour q' in F_j meets x , since the witnesses at x miss q' . If an edge f of F_j of another colour avoided x , goodness would give an edge e of F_i of a colour other than c avoiding x and f , and the blocks of $\{x, c, q'\}$, e and f would be three disjoint triples in place of M_i and M_j . So (x, q') is open in F_j . Distinct q' give distinct pairs, and each colour of R that is not good spoils at most one. So at least $t + g_x - S_x$ open pairs come from x . Summing over the x with $g_x > 0$, and using $S_x \geq t$ for the rest, gives $tu + G - S$. $\square$

Three anchors.

- A properly three-coloured K_4 on a set Y has $G = 3(u - 4)$: for $x \notin Y$ all three colours are missing, every pair is forbidden, and every colour is good, since the other two colours form a four-cycle on Y and a pair f outside F_i meets Y in at most one point.
- A spanning one-coloured large F_i has $G = 2u$: for every x its two unused colours are missing, forbidden together, and good, since a matching of size four leaves an edge avoiding x and f .
- The graph Y_k , with edges pa and pb of two colours and a third colour made of ab and k edges at p , has $G = 2(u - 3)$.

Lemma 13 (finite classification; code/verify_anchor_count.py). *Let $u = 11$ and $1 \leq d \leq 4$. Every local graph with more than one colour has at most $12 - S_j$ open pairs, and every one with $S_j \geq 11$ has anchor value at least 16. A one-coloured graph of support s has $11 - s$ open pairs, plus two for each vertex common to all its edges.*

The one-coloured count is direct. The script checks every multicoloured local graph on eleven vertices that satisfies (2.2) and Lemma 4: those with $S_j \geq 11$ are K_4 and Y_4 , and at $d = 4$ also Y_5 , with anchor values 21, 16 and 16.

Proposition 2. *In the equality setting, $u = 11$ with $1 \leq d \leq 4$ is impossible.*

Proof. Here $t = 10 - d$, every $S_j \leq 12$ by Lemma 5, and $S \geq 11t$. So $\sum_j (12 - S_j) \leq t \leq 9$, and some $S_j \geq 11$. That F_j is an anchor with $G \geq 16$: by Lemma 13 if it has several colours; otherwise it spans U with at most $(11 + d)/2 \leq 7$ edges, so its matching number is at least four and $G = 22$. By Lemma 12 the other local graphs carry at least $11t + 16 - S$ open pairs.

By Lemma 13 a local graph carries at most $12 - S_j$ open pairs, except a one-coloured one whose edges share a vertex. A star with k leaves has $12 - S_j = 11 - k$ and carries one more; a single edge or an empty graph would need $12 - S_j \geq 10 > t$. A star needs $11 - k \leq t$ and, by (2.2), $k \leq d + 1$, so $k = d + 1$ and it uses the whole budget; at most one occurs. So the local graphs carry at most $12t - S + 1$ open pairs, and $11t + 16 \leq 12t + 1$ would need $t \geq 15$. $\square$

4.4 Proof of Theorem 2

Suppose $r \geq 16$, $|E(H)| = 3r - 3$ and $\tau(H) = r$. By §4.2, $u \leq 11$, and $r = 2u - 2 - d$ by (3.1).

- If $u \leq 9$, then $r \leq 16 - d$, so $r = 16$, $u = 9$, $d = 0$, excluded by Proposition 1.
- If $u = 11$, then $d = 20 - r \leq 4$: $d = 0$ is excluded by Proposition 1, and $1 \leq d \leq 4$ by Proposition 2.

So $|E(H)| \geq 3r - 2$ by Corollary 1. $\square$

4.5 The remaining finite case

At $r = 15$ the parameter choices are $(u, d, t) = (9, 1, 7)$ and $(11, 5, 5)$. Proposition 1 excludes the first. The second remains open: at $d = 5$ multicoloured stars can have support $u = 11$ with zero anchor value, and the count of Lemma 13 no longer applies. So the argument stops at $r \geq 16$ and does not settle $7 \leq r \leq 15$.

5. Verification

- Theorem 1 on every configuration with at most six hyperedges, and at low degree up to nine, and the sharpness family: `code/verify_residual_bound.py`.
- Lemmas 1, 3 and 5, the $d = 0$ structure of §2.3, and the $u = 13$, $d = 5$ step: `code/verify_local_graphs.py`.
- Lemmas 10 and 11 on 8, 9 and 10 vertices: `code/verify_rainbow_lemmas.py` and `code/verify_rainbow.c`.
- Lemma 13, the anchor values, and the failure at $d = 5$ in §4.5: `code/verify_anchor_count.py`.
- The arithmetic of §2.3, §3.4 and §4: `code/verify_p5_arithmetic.py`.
- The 21-hyperedge family for $r = 7$ in §1: `code/verify_witness_r7_21.py` (standard library only); its generator is `code/search_symmetric_family.py`.

The 21-hyperedge family for $r = 7$, as an incidence matrix in the format of [4]: one row for each vertex $1, \dots, 35$ in order, one column for each hyperedge $1, \dots, 21$ in order (the cards of `data/witness_r7_21.json`), and 1 where the vertex lies in the hyperedge. Each column has seven ones; the rows have weight 6 (vertices 1–7), 4 (vertices 8–28) and 3 (vertices 29–35). `code/verify_incidence_matrices.py` checks that it is exactly the matrix of the data file.

```

111001010000001000000
011100101000000100000
101110000100000010000
010111000010000001000
001011100001000000100
100101100000100000010
110010100000010000001
100000000001000101000
010000000000100010100
001000000000010001010
000100010000000000101
000010001000001000010
000001000100000100001
000000100010001010000
010000000001000010010
001000000000100001001
000100000000011000100
000010010000000100010
000001001000000010001
000000100100001001000
100000000010000100100
000010011000100000000
000001001100010000000

```

```

000000110110000000000
100000001011000000000
010000000101100000000
001000000010110000000
000100010001010000000
000000010000000011000
000000001000000001100
000000000100000000110
000000000010000000011
000000000001000000001
0000000000001001000001
0000000000000101100000
0000000000000010110000

```

The runs are recorded in `results/p5_checks.txt`. The steps that are not finite have been read but not independently reviewed.

Acknowledgements and research status

AI assistance (ChatGPT, OpenAI; Claude, Anthropic) was used for algebra, drafting, programming and internal review. The author is responsible for the mathematical claims. No priority is claimed.

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